Question 1	Question 2	Question 3	Question 4	Sum	Final score

'Second compitino' and written exam ('primo appello') of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica - 31 January 2013.

SURNAME.................... NAME.................... MATRICOLA....................

PLEASE NOTE. During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered. Marked tests will be handed out in room 2BC60 on 5 February 2013 at 14:30.

Duration: 150 minutes

Please tick one of the two options below:

☐ Second 'compitino': questions 3 and 4. Time 80 Minutes.
☐ Written exam: all questions. Time 150 Minutes.

Question 1.

(i) Give the definition of convolution and mollifier.
(ii) Prove that convolution is associative, i.e. $(f * g) * h = f * (g * h)$.
(iii) State the generalized Young's inequality.

Answer:

Question 2.

(i) Let $\Omega = B_{\mathbb{R}^N}(0,1)$ (the unitary ball centered at zero in $\mathbb{R}^N$). Assume that $N \geq 2$. Let f be the function defined almost everywhere in Ω by the formula $f(x) = |x|^\mu$, for all $x \in \Omega \setminus \{0\}$ where $\mu \in \mathbb{R}$. Let $l \in \mathbb{N}$. Find all values of $\mu \in \mathbb{R}$ and $p \in [1, \infty[$ such that $f \in W^{l,p}(\Omega)$. (Detailed motivations are required.)

(ii) Give the definition of the space $W_0^{l,p}(\Omega)$.

(iii) Is it true that if Ω is a bounded open set in $\mathbb{R}^N$ of class C^1 and $f \in W_0^{1,p}(\Omega)$ then $f \in L^\infty(\Omega)$? If yes, prove it. If not, give a counterexample.

Answer:

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Question 3.

(i) State the Sobolev's Embedding Theorem part 2.

(ii) In the case $\Omega = \mathbb{R}^N$ prove that the Sobolev's exponent q^* in (i) cannot be improved, i.e. prove that the space $W^{l,p}(\mathbb{R}^N)$ is not embedded into $L^q(\mathbb{R}^N)$ with $q > q^*$.

(iii) State in detail the theorem concerning the decomposition of open sets satisfying the cone condition into open sets which are star-shaped with respect to balls.

Answer:

Question 4.

(i) Give the definition of compact operator between normed spaces.

(ii) State the Rellich-Kondrakov Theorem.

(iii) Assume that Ω is a bounded open set of class C^1 and $1 < p < \infty$. By using (ii) prove that the Sobolev space $W^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$ for any $q < p$.

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